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A note on the higher-order Frobenius-Euler polynomials and Sheffer sequences

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Abstract

In this paper, we investigate some properties of polynomials related to Sheffer sequences. Finally, we derive some identities of higher-order Frobenius-Euler polynomials.

1 Introduction

Let $\lambda\ (\neq 1) \in \mathbb{C}$. The higher-order Frobenius-Euler polynomials are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} e^{xt} = e^{H^{(\alpha)}(x|\lambda)t} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda)\frac{t^n}{n!} \quad \text{(see [1–17])} \tag{1}$$

with the usual convention about replacing $(H^{(\alpha)}(x|\lambda))^n$ by $H_n^{(\alpha)}(x|\lambda)$. In the special case, $x = 0$, $H_n^{(\alpha)}(0|\lambda) = H_n^{(\alpha)}(\lambda)$ are called the nth Frobenius-Euler numbers of order α $(\in \mathbb{R})$.

From (1) we have

$$H_n^{(\alpha)}(x|\lambda) = \sum_{l=0}^{n}\binom{n}{l}H_{n-l}^{(\alpha)}(\lambda)x^l = \sum_{l=0}^{n}\binom{n}{l}H_{n-l}^{(\alpha)}(\lambda)x^{n-l} \quad \text{(see [6])}. \tag{2}$$

By (2) we get

$$\frac{\partial}{\partial x}H_n^{(\alpha)}(x|\lambda) = nH_{n-1}^{(\alpha)}(x|\lambda), \qquad H_n^{(0)}(x|\lambda) = x^n \quad \text{for } n \in \mathbb{Z}_+. \tag{3}$$

It is not difficult to show that

$$H_n^{(\alpha)}(x+1|\lambda) - \lambda H_n^{(\alpha)}(x|\lambda) = (1-\lambda)H_n^{(\alpha-1)}(x|\lambda) \quad \text{(see [1–17])}. \tag{4}$$

Let us define the λ-difference operator $\triangle_\lambda$ as follows:

$$\triangle_\lambda f(x) = f(x+1) - \lambda f(x). \tag{5}$$

From (5) we can derive the following equation:

$$\begin{aligned}\triangle_\lambda^n f(x) = \underbrace{\triangle_\lambda \cdots \triangle_\lambda}_{n\text{-times}} f(x) &= \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} f(x+k) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{k} f(x+n-k). \qquad (6)\end{aligned}$$

As is well known, the Stirling numbers $S(l,n)$ of the second kind are defined by the generating function to be

$$\left(e^t - 1\right)^n = n! \sum_{l=0}^{\infty} S(l,n) \frac{t^l}{l!} \quad \text{(see [5, 6, 11])} \qquad (7)$$

and

$$\left(e^t - 1\right)^n = \sum_{l=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} m^l \right) \frac{t^l}{l!}. \qquad (8)$$

By (7) and (8), we get

$$S(l,n) = \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} m^l = \frac{\triangle^n 0^l}{n!} \quad \text{(see [11])}, \qquad (9)$$

where $\triangle f(x) = f(x+1) - f(x)$.

Now, we consider the λ-analogue of the Stirling numbers of the second kind as follows:

$$\left(e^t - \lambda\right)^n = n! \sum_{l=0}^{\infty} S(l,n|\lambda) \frac{t^l}{l!} \qquad (10)$$

and

$$\left(e^t - \lambda\right)^n = \sum_{l=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} (-\lambda)^{n-m} m^l \right) \frac{t^l}{l!}. \qquad (11)$$

From (10) and (11), we have

$$S(l,n|\lambda) = \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (-\lambda)^{n-m} m^l = \frac{1}{n!} \triangle_\lambda^n 0^l \qquad (12)$$

and

$$S(l,n|\lambda) = 0 \quad \text{for } n > l. \qquad (13)$$

From (4) and (5), we have

$$\triangle_\lambda H_n^{(\alpha)}(x|\lambda) = (1-\lambda) H_n^{(\alpha-1)}(x|\lambda). \qquad (14)$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{n=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C} \right\}.$$

$\mathbb{P}$ indicates the algebra of polynomials in the variable x over $\mathbb{C}$, and $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$ (see [5, 11]). In [11], $\langle L|p(x)\rangle$ denotes the action of the linear functional L on a polynomial $p(x)$, and we remind that the vector space structure on $\mathbb{P}^*$ is defined by

$$\begin{aligned}
&\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \\
&\langle cL|p(x)\rangle = c\langle L|p(x)\rangle,
\end{aligned}$$

where c is a complex constant.

The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F} \tag{15}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad \text{for all } n \in \mathbb{Z}_+ \text{ (see [11]).} \tag{16}$$

From (15) and (16), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad \text{(see [5, 11]).} \tag{17}$$

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$. From (17) we have

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle \quad \text{for all } n \in \mathbf{Z}_+. \tag{18}$$

By (18) we get $L = f_L(t)$. It is known in [11] that the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We will call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra (see [5, 11]).

The order $O(f(t))$ of the nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ has $O(f(t)) = 1$ is called a delta series and a series $f(t)$ has $O(f(t)) = 0$ is called an invertible series (see [5, 11]). By (16) and (17), we get $\langle e^{yt}|x^n\rangle = y^n$, and so $\langle e^{yt}|p(x)\rangle = p(y)$ (see [5, 11]). For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k. \tag{19}$$

Let $f_1(t), f_2(t), \ldots, f_n(t) \in \mathcal{F}$. Then we see that

$$\begin{aligned} &\langle f_1(t) f_2(t) \cdots f_n(t) | x^n \rangle \\ &\quad = \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1, \ldots, i_m} \langle f_1(t) | x^{i_1} \rangle \cdots \langle f_m(t) | x^{i_m} \rangle, \end{aligned} \tag{20}$$

where $\binom{n}{i_1,\ldots,i_m} = \frac{n!}{i_1! \cdots i_m!}$ (see [5, 11]).

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, it is easy to show that

$$\langle f(t) g(t) | p(x) \rangle = \langle f(t) | g(t) p(x) \rangle = \langle g(t) | f(t) p(x) \rangle \quad \text{(see [11])}. \tag{21}$$

From (19), we can derive the following equation:

$$p^{(k)}(0) = \langle t^k | p(x) \rangle \quad \text{and} \quad \langle 1 | p^{(k)}(x) \rangle = p^{(k)}(0). \tag{22}$$

By (22) we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [5, 11])}. \tag{23}$$

Thus, from (23) we have

$$e^{yt} p(x) = p(x+y). \tag{24}$$

Let $S_n(x)$ be polynomials in the variable x with degree n, and let $f(t)$ be a delta series and $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials with $\langle g(t) f(t)^k | S_n(x) \rangle = n! \delta_{n,k}$ $(n, k \geq 0)$, where $\delta_{n,k}$ is the Kronecker symbol. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequence for $f(t)$. If $S_n(x) \sim (g(t), t)$, then $S_n(x)$ is called the Appell sequence for $g(t)$ (see [5, 11]). For $p(x) \in \mathbb{P}$, the following equations (25)-(27) are known in [5, 11]:

$$\left\langle \frac{e^{yt}-1}{t} \middle| p(x) \right\rangle = \int_0^y p(u)\, du, \tag{25}$$

$$\langle f(t) | x p(x) \rangle = \langle \partial_t f(t) | p(x) \rangle = \langle f'(t) | p(x) \rangle, \tag{26}$$

and

$$\langle e^{yt} - 1 | p(x) \rangle = p(y) - p(0). \tag{27}$$

For $S_n(x) \sim (g(t), f(t))$, we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | S_k(x) \rangle}{k!} g(t) f(t)^k, \quad h(t) \in \mathcal{F}, \tag{28}$$

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k | p(x) \rangle}{k!} S_k(x), \quad p(x) \in \mathbb{P}, \tag{29}$$

$$f(t)S_n(x) = nS_{n-1}(x), \qquad \langle f(t)|p(\alpha x)\rangle = \langle f(\alpha t)|p(x)\rangle, \tag{30}$$

$$\frac{1}{g(\bar{f}(t))}e^{y\bar{f}(t)} = \sum_{k=0}^{\infty}\frac{S_k(y)}{k!}t^k \quad \text{for all } y \in \mathbb{C}, \tag{31}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

$$S_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}P_k(y)S_{n-k}(x) = \sum_{k=0}^{n}\binom{n}{k}P_k(x)S_{n-k}(y), \tag{32}$$

where $P_k(y) = g(t)S_k(y) \sim (1, f(t))$ (see [5, 11]).

In contrast to the higher-order Euler polynomials, the more general higher-order Frobenius-Euler polynomials have never been studied in the context of umbral algebra and umbral calculus.

In this paper, we investigate some properties of polynomials related to Sheffer sequences. Finally, we derive some identities of higher-order Frobenius-Euler polynomials.

2 Associated sequences

Let $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then, for $n \geq 1$, we note that

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x) \quad \text{(see [11])}. \tag{33}$$

Let us take $f(t) = e^{at} - 1$ ($a \neq 0$). Then we see that $f'(t) = ae^{at}$, $\bar{f}(t) = a^{-1}\log(t+1)$.

From (27), we can derive the associated sequence $p_n(x)$ for $f(t) = e^{at} - 1$ as follows:

$$\begin{aligned} p_n(y) &= \langle e^{yt}|p_n(x)\rangle = \langle e^{y\bar{f}(t)}|x^n\rangle = \langle e^{\frac{y}{a}\log(t+1)}|x^n\rangle \\ &= \langle (t+1)^{\frac{y}{a}}|x^n\rangle = \sum_{k=0}^{\infty}\binom{\frac{y}{a}}{k}\langle t^k|x^n\rangle \\ &= \sum_{k=0}^{n}\binom{\frac{y}{a}}{k}n!\delta_{n,k} = \binom{\frac{y}{a}}{n}n! = \left(\frac{y}{a}\right)_n, \end{aligned} \tag{34}$$

where $(a)_n = a(a-1)\cdots(a-n+1) = \Pi_{i=0}^{n-1}(a-i)$.

Therefore, by (34) we obtain, for $n \in \mathbb{Z}_+$,

$$p_n(x) = \left(\frac{x}{a}\right)_n \sim (1, e^{at} - 1).$$

We get the following:

$$\begin{aligned} p_{n+1}(x) &= x\big(f'(t)\big)^{-1}p_n(x) = a^{-1}xe^{-at}p_n(x) \\ &= \left(\frac{x}{a}\right)\left(\frac{x-a}{a}\right)_n = \left(\frac{x}{a}\right)p_n(x-a). \end{aligned} \tag{35}$$

From (35), we can derive the equation

$$\begin{aligned} p_{n+1}(x) &= \left(\frac{x}{a}\right) p_n(x-a) = \left(\frac{x}{a}\right)\left(\frac{x-a}{a}\right) p_n(x-2a) \\ &= \left(\frac{x}{a}\right)\left(\frac{x-a}{a}\right)\left(\frac{x-2a}{a}\right) p_n(x-3a) = \cdots \\ &= \left(\frac{x}{a}\right)\left(\frac{x}{a}-1\right)\left(\frac{x}{a}-2\right)\cdots\left(\frac{x}{a}-n\right). \end{aligned} \tag{36}$$

By (19) we get

$$\left(\frac{x}{a}\right)_n = \sum_{k=0}^{\infty} \frac{\langle t^k | (\frac{x}{a})_n \rangle}{k!} x^k = \sum_{k=0}^{n} \frac{S_1(n,k)}{a^k} x^k \tag{37}$$

and

$$\frac{1}{k!}\left\langle t^k \middle| \left(\frac{x}{a}\right)_n \right\rangle = \frac{S_1(n,k)}{a^k}, \tag{38}$$

where $S_1(n,k)$ is the Stirling numbers of the first kind.

Therefore, by (37) and (38), we obtain the following theorem.

Lemma 1 *For* $n, k \geq 0$, *we have*

$$\frac{\langle t^k | (\frac{x}{a})_n \rangle}{k!} = \frac{S_1(n,k)}{a^k}.$$

From (31) we note that

$$\sum_{k=0}^{\infty} \frac{(\frac{x}{a})_k}{k!} t^k = e^{\frac{x}{a}\log(1+t)} = (t+1)^{\frac{x}{a}}. \tag{39}$$

And by (32) we get

$$\left(\frac{x+y}{a}\right)_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{a}\right)_k \left(\frac{y}{a}\right)_{n-k}. \tag{40}$$

As is well known, the nth Frobenius-Euler polynomials are defined by the generating function to be

$$\frac{1-\lambda}{e^t-\lambda} e^{xt} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{t^n}{n!}. \tag{41}$$

Thus, by (42) we see that $H_n(x|\lambda) \sim (\frac{e^t-\lambda}{1-\lambda}, t)$. So, we note that

$$\frac{e^t-\lambda}{1-\lambda} H_n(x|\lambda) \sim (1,t). \tag{42}$$

It is easy to show that $x^n \sim (1, t)$ (see Eq. (17)). Thus, from (42) we have

$$\begin{aligned} x^n &= x\left(\frac{t}{t}\right)^n x^{-1}\left(\frac{e^t-\lambda}{1-\lambda}H_n(x|\lambda)\right) = \frac{1}{1-\lambda}\left(e^t-\lambda\right)H_n(x|\lambda) \\ &= \frac{1}{1-\lambda}\left(H_n(x+1|\lambda)-\lambda H_n(x|\lambda)\right). \end{aligned} \tag{43}$$

3 Frobenius-Euler polynomials of order α

From (1) and (31), we note that

$$H_n^{(\alpha)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^\alpha, t\right) \tag{44}$$

and

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^\alpha x^n = H_n^{(\alpha)}(x|\lambda) \quad \text{for all } n \geq 0. \tag{45}$$

From (32), we have

$$\begin{aligned} H_n^{(\alpha)}(x+y) &= \sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(y|\lambda)x^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x|\lambda)y^{n-k}. \end{aligned} \tag{46}$$

Let us take the operator $(\frac{1-\lambda}{e^t-\lambda})^\beta$ on both sides of (46).

Then we have

$$\begin{aligned} \left(\frac{1-\lambda}{e^t-\lambda}\right)^\beta H_n^{(\alpha)}(x+y|\lambda) &= \left(\frac{1-\lambda}{e^t-\lambda}\right)^\beta\left(\frac{1-\lambda}{e^t-\lambda}\right)^\alpha (x+y)^n \\ &= \left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha+\beta}(x+y)^n = H_n^{(\alpha+\beta)}(x+y|\lambda) \end{aligned} \tag{47}$$

and by (46) we get

$$\begin{aligned} H_n^{(\alpha+\beta)}(x+y|\lambda) &= \sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(y|\lambda)\left(\frac{1-\lambda}{e^t-\lambda}\right)^\beta x^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(y|\lambda)H_{n-k}^{(\beta)}(x|\lambda). \end{aligned} \tag{48}$$

Therefore, by (48) we obtain the following proposition.

Proposition 2 *For $\alpha, \beta \in \mathbb{C}$ and $n \geq 0$, we have*

$$\begin{aligned} H_n^{(\alpha+\beta)}(x+y|\lambda) &= \sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x|\lambda)H_{n-k}^{(\beta)}(y|\lambda) \\ &= \sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(y|\lambda)H_{n-k}^{(\beta)}(x|\lambda). \end{aligned}$$

Thus, we have

$$e^{xt} = \sum_{n=0}^{\infty} H_n^{(0)}(x|\lambda)\frac{t^n}{n!}. \tag{49}$$

Thus, by (49) we get

$$H_n^{(0)}(x|\lambda) = x^n. \tag{50}$$

Let us take $\beta = -\alpha$. Then, from Proposition 2, we have

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(x|\lambda)H_k^{(-\alpha)}(y|\lambda)\\ &= \sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(y|\lambda)H_k^{(-\alpha)}(x|\lambda).\end{aligned} \tag{51}$$

Therefore, by (51) we obtain the following corollary.

Corollary 3 *For $n \geq 0$, we have*

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(x|\lambda)H_k^{(-\alpha)}(y|\lambda)\\ &= \sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(y|\lambda)H_k^{(-\alpha)}(x|\lambda).\end{aligned}$$

In the special case, $y = 0$, we have

$$x^n = \sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(x|\lambda)H_k^{(-\alpha)}(\lambda).$$

Let $\alpha \in \mathbb{N}$. We get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_n^{(-\alpha)}(\lambda)}{n!}t^n &= \left(\frac{e^t-\lambda}{1-\lambda}\right)^{\alpha}\\ &= \frac{1}{(1-\lambda)^{\alpha}}\sum_{l=0}^{\alpha}\binom{\alpha}{l}(-1)^{\alpha-l}\lambda^{\alpha-l}e^{lt}\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{(1-\lambda)^{\alpha}}\sum_{l=0}^{\alpha}\binom{\alpha}{l}(-1)^{\alpha-l}\lambda^{\alpha-l}l^n\right)\frac{t^n}{n!}.\end{aligned} \tag{52}$$

Thus, from (52) we have

$$\begin{aligned}H_n^{(-\alpha)}(\lambda) &= \frac{1}{(1-\lambda)^{\alpha}}\sum_{l=0}^{\alpha}\binom{\alpha}{l}(-1)^{\alpha-l}\lambda^{\alpha-l}l^n\\ &= \frac{1}{(1-\lambda)^{\alpha}}\triangle_{\lambda}^{\alpha}0^n = \frac{\alpha!}{(1-\lambda)^{\alpha}}\frac{\triangle_{\lambda}^{\alpha}0^n}{\alpha!} = \frac{\alpha!}{(1-\lambda)^{\alpha}}S(n,\alpha|\lambda).\end{aligned} \tag{53}$$

Therefore, by (51), (52) and (53), we obtain the following theorem.

Theorem 4 *For $\alpha \in \mathbb{N}$ and $n \geq 0$, we have*

$$\frac{(1-\lambda)^\alpha}{\alpha!}x^n = \sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(x|\lambda)S(k,\alpha|\lambda).$$

From (19), we have

$$\begin{aligned} x^n &= \sum_{k=0}^{n}\frac{\langle(\frac{e^t-\lambda}{1-\lambda})^\alpha t^k|x^n\rangle}{k!}H_k^{(\alpha)}(x|\lambda) \\ &= \sum_{k=0}^{n}\frac{1}{k!}\left\langle\left(\frac{e^t-\lambda}{1-\lambda}\right)^\alpha\middle|t^k x^n\right\rangle H_k^{(\alpha)}(x|\lambda) \\ &= \sum_{k=0}^{n}\binom{n}{k}\left\langle\left(\frac{e^t-\lambda}{1-\lambda}\right)^\alpha\middle|x^{n-k}\right\rangle H_k^{(\alpha)}(x|\lambda) \end{aligned} \tag{54}$$

and

$$\begin{aligned} \left\langle\left(\frac{e^t-\lambda}{1-\lambda}\right)^\alpha\middle|x^{n-k}\right\rangle &= \sum_{j=0}^{\infty}\frac{H_j^{(-\alpha)}(\lambda)}{j!}\langle t^j|x^{n-k}\rangle \\ &= \sum_{j=0}^{\infty}\frac{H_j^{(-\alpha)}(\lambda)}{j!}\delta_{n-k,j}(n-k)! \\ &= H_{n-k}^{(-\alpha)}(\lambda). \end{aligned} \tag{55}$$

By (54) and (55), we also get

$$x^n = \sum_{k=0}^{n}\binom{n}{k}H_k^{(\alpha)}(x|\lambda)H_{n-k}^{(-\alpha)}(\lambda).$$

4 Further remark

Let us take $a = 1$ in (34). Then we have $(x)_n \sim (1, f(t) = e^t - 1)$, $x^n \sim (1, g(t) = t)$.

For $n \geq 1$, by (33) we get

$$\begin{aligned} x^n &= x\left(\frac{e^t-1}{t}\right)^n x^{-1}(x)_n = x\left(\frac{e^t-1}{t}\right)^n (x-1)_{n-1} \\ &= x\sum_{l=0}^{\infty}\frac{n!}{(l+n)!}S_2(l+n,n)t^l(x-1)_{n-1}, \end{aligned} \tag{56}$$

where $S_2(m,n)$ is the Stirling numbers of the second kind.

From (56) we have

$$(x+1)^{n+1} = (x+1)\sum_{l=0}^{n}\frac{(n+1)!}{(l+n+1)!}S_2(l+n+1,n+1)t^l(x)_n, \tag{57}$$

where

$$(x)_n = \sum_{l=0}^{n}S_1(n,k)x^k. \tag{58}$$

Thus, by (58) we get

$$t^l(x)_n = \frac{d^l}{dt^l}(x)_n = \sum_{k=l}^{n} S_1(n,k)(k)_l x^{k-l}. \tag{59}$$

From (57) and (59), we can derive the following equation:

$$\begin{aligned}(x+1)^n &= \sum_{l=0}^{n} \frac{(n+1)!}{(l+n+1)!} S_2(l+n+1,n+1) \sum_{k=l}^{n} S_1(n,k)(k)_l x^{k-l} \\ &= \sum_{l=0}^{n}\sum_{m=0}^{n-l} \frac{\binom{l+m}{l}}{\binom{n+l+1}{l}} S_2(l+n+1,n+1)S_1(n,l+m)x^m \\ &= \sum_{m=0}^{n}\left\{\sum_{l=0}^{n-m} \frac{\binom{l+m}{l}}{\binom{n+l+1}{l}} S_2(l+n+1,n+1)S_1(n,l+m)\right\}x^m \end{aligned} \tag{60}$$

and

$$(x+1)^n = \sum_{l=0}^{n} \binom{n}{l} x^l. \tag{61}$$

Therefore, by (60) and (61), we obtain the following lemma.

Lemma 5 *For $0 \le m \le n$, we have*

$$\binom{n}{m} = \sum_{l=0}^{n-m} \frac{\binom{l+m}{l}}{\binom{n+l+1}{l}} S_2(l+n+1,n+1)S_1(n,l+m).$$

Let $\alpha = 1$. Then we write $H_n^{(1)}(x|\lambda) = H_n(x|\lambda)$. From (34), we note that

$$(x)_n \sim \left(1, e^t - 1\right). \tag{62}$$

Thus, by (19) and (62), we get

$$\begin{aligned} H_n(x|\lambda) &= \sum_{k=0}^{\infty} \frac{\langle (e^t-1)^k | H_n(x|\lambda)\rangle}{k!}(x)_k \\ &= H_n(\lambda) + \sum_{k=1}^{\infty} \frac{\langle (e^t-1)^k | H_n(x|\lambda)\rangle}{k!}(x)_k. \end{aligned} \tag{63}$$

For $k \ge 1$, we have

$$\left\langle \left(e^t-1\right)^k | H_n(x|\lambda)\right\rangle = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(\lambda) \left\langle \left(e^t-1\right)^k | x^l\right\rangle. \tag{64}$$

From (9), we have

$$\frac{1}{k!}\left\langle \left(e^t-1\right)^k | x^l\right\rangle = S_2(l,k). \tag{65}$$

Therefore, by (63), (64) and (65), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$\begin{aligned} H_n(x|\lambda) &= H_n(\lambda) + \sum_{k=1}^{n} \sum_{l=0}^{n} \binom{n}{l} H_{n-l}(\lambda) S_2(l,k)(x)_k \\ &= H_n(\lambda) + \sum_{k=1}^{n} \sum_{l=0}^{n} \sum_{m=0}^{k} \binom{n}{l} H_{n-l}(\lambda) S_2(l,k) S_1(k,m) x^m. \end{aligned}$$

From the recurrence formula of the Appell sequence, we note that

$$\begin{aligned} H_{n+1}^{(\alpha)}(x|\lambda) &= \left(x - \alpha \frac{e^t}{e^t - \lambda}\right) H_n^{(\alpha)}(x|\lambda) = x H_n^{(\alpha)}(x|\lambda) - \frac{\alpha e^t}{e^t - \lambda} H_n^{(\alpha)}(x|\lambda) \\ &= x H_n^{(\alpha)}(x|\lambda) - \frac{\alpha}{(1-\lambda)} \frac{1-\lambda}{e^t - \lambda} H_n^{(\alpha)}(x+1) \\ &= x H_n^{(\alpha)}(x|\lambda) - \frac{\alpha}{(1-\lambda)} H_n^{(\alpha+1)}(x+1). \end{aligned} \tag{66}$$

Therefore, by (66) we obtain the following theorem.

Theorem 7 *For $n \geq 0$, we have*

$$H_{n+1}^{(\alpha)}(x|\lambda) = x H_n^{(\alpha)}(x|\lambda) - \frac{\alpha}{(1-\lambda)} H_n^{(\alpha+1)}(x+1).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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